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Several differential equations for Genocchi polynomials combined with trigonometric functions

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Abstract. In this paper, we find q-differential equations for polynomials that appear by combining trigonometric functions with Genocchi polynomials. Since Genocchi polynomials (QSG and QCG) come in two forms, we can identify interesting differential equations for the variables x and y.

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1. Introduction

This section briefly outlines the essential definitions and theorems required for understanding this study. For $q \in \mathbb{R} - \{1\}$, the q-number is defined as:

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

In the definition of the q-number, it noted that $\lim_{q\to 1}[n]_q = n$, see [2], [3], [9]. Moreover, for $k \in \mathbb{Z}$, $[k]_q$ is referred to as a q-integer.

The q-numbers introduced by Jackson [3] have led to expanded theories that intersect with established fields, see, [1], [2], [8], [9]. The q-Gaussian binomial coefficients ([4]) are defined as

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[m-r]_q![r]_q!},$$

Here, m and r denote non-negative integers.

Note that $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[0]_q! = 1$.

Definition 1.1. Let x be any complex numbers with $|x| < 1$. Then, two forms of q-exponential functions ([1], [2]) can be expressed as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad E_q(x), \\ = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}.$$

It is noted that $\lim_{q\to 1} e_q(x) = e^x$ and $e_q(x)E_q(-x) = 1$.

Definition 1.2. The q-derivative of a function f with respect to x is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \text{for} \quad x \neq 0,$$

and $D_q f(0) = f'(0)$, see, [7], [9].

We can prove that f is differentiable at zero, and it is clear that $D_q x^n = [n]_q x^{n-1}$. Because the polynomials covered in this study deal with multiple variables, we use the derivative with respect to x, y, and t, which are expressed as $D_{q,x}$, $D_{q,y}$, and $D_{q,t}$, respectively.

Definition 1.3. The generating function for the q-Genocchi numbers and polynomials ([5], [6]) are

$$\sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}, \\ \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1} e_q(tx), \quad \text{respectively.}$$

For $q \to 1$ in Definition 1.3., we can find the Genocchi numbers G_n and polynomials $G_n(x)$.

In [5], the authors introduced new Genocchi polynomials (sine Genocchi polynomials and cosine Genocchi polynomials) by replacing x with complex numbers and studied several properties thereof.

Furthemore, [6] combines the polynomials discussed in [5] with q-numbers to construct an Genocchi polynomial that incorporates q-trigonometric functions. The study also reveals associated properties and symmetrical structures. Specifically, the authors pinpoint approximate roots that fluctuate based on the value of q and present a visual representation of these roots.

Definition 1.4. The generating function for the q-SINE Genocchi (QSG) and q-COSINE Genocchi (QCG) polynomials are

$$\sum_{n=0}^{\infty} G_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx)\mathrm{SIN}_q(ty),$$
$$\sum_{n=0}^{\infty} G_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx)\mathrm{COS}_q(ty),$$

respectively, see [5].

Theorem 1.5 [7]. *Let k be a non-negative integer. Then, the following relations can be formulated:*

$$(i) \quad S_{n-k,q}(x,y) = \frac{[n-k]_q!}{[n]_q!} D_{q,x}^{(k)} S_{n,q}(x,y).$$
$$(ii) \quad C_{n-k,q}(x,y) = \frac{[n-k]_q!}{[n]_q!} D_{q,x}^{(k)} C_{n,q}(x,y).$$
□

Theorem 1.6 [7]. *Let k be a non-negative integer. Then, the following is valid:*

$$(i) \quad D_{q,y}^{(k)} S_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}} \dfrac{[n]_q!}{[n-k]_q!} S_{n-k,q}(x,q^k y), & \text{if } k\text{: even}, \\ \\ (-1)^{\frac{k-1}{2}} \dfrac{[n]_q!}{[n-k]_q!} C_{n-k,q}(x,q^k y), & \text{if } k\text{: odd}. \end{cases}$$

$$(ii) \quad D_{q,y}^{(k)} C_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}} \dfrac{[n]_q!}{[n-k]_q!} C_{n-k,q}(x, q^k y), & \text{if } k\text{: even}, \\ (-1)^{\frac{k+1}{2}} \dfrac{[n]_q!}{[n-k]_q!} S_{n-k,q}(x, q^k y), & \text{if } k\text{: odd}. \end{cases}$$

□

The organization of this study is as follows:

Section 2 elaborates on the q-difference equations associated with the QSG polynomial, drawing upon the theorems established in the preceding section. We identify multiple q-difference equations that vary both by the type of polynomial and the variables.

2. Main results

In this Section, we use the Theorems 2.1. and 2.2. to verify the q-difference equations associated with QSG and QCG polynomials. The q-difference equations that vary based on the variables are shown to have QSG and QCG polynomials as solutions.

Theorem 2.1. *For $k \in$ non-negative integer, we have the following relations with ${}_CG_{n,q}(x,y)$ and ${}_SG_{n,q}(x,y)$:*

$$(i) \quad D_{q,x}^{(k)} {}_C G_{n,q}(x,y) = \frac{[n]_q!}{[n-k]_q!} {}_C G_{n-k,q}(x,y),$$

$$(ii) \quad D_{q,x}^{(k)} {}_S G_{n,q}(x,y) = \frac{[n]_q!}{[n-k]_q!} {}_S G_{n-k,q}(x,y).$$

Proof. (i) Using the q-derivative in ${}_CG_{n,q}(x,y)$ about x, we get:

$$\begin{aligned} & D_{q,x}^{(1)} \sum_{n=0}^{\infty} {}_C G_{n,q}(x,y) \frac{t^n}{[n]_q!} \\ &= t \sum_{n=0}^{\infty} {}_C G_{n,q}(x,y) \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} [n]_q {}_C G_{n-1,q}(x,y) \frac{t^n}{[n]_q!}. \end{aligned} \tag{1}$$

After comparing the coefficients of t^n in Equation (1), we can formulate:

$$D^{(1)}_{q,x}\,{}_CG_{n,q}(x,y) = [n]_q\,{}_CG_{n-1,q}(x,y) = \frac{[n]_q!}{[n-1]_q!}\,{}_CG_{n-1,q}(x,y).$$

Via induction, we obtain Theorem 2.1 (i).

(ii) If we apply the proof of (i) of the Theorem 2.1 similarly to ${}_SG_{n,q}(x,y)$, we can derive (ii) of the theorem; hence, the proof process is omitted. □

Theorem 2.2. *Let k be a non-negative integer. Then, the following hold:*

$$(i)\quad D^{(k)}_{q,y}\,{}_C\,G_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}}\dfrac{[n]_q!}{[n-k]_q!}\,{}_CG_{n-k,q}(x,q^ky), & \text{if } k\text{: even,} \\[2ex] (-1)^{\frac{k+1}{2}}\dfrac{[n]_q!}{[n-k]_q!}\,{}_SG_{n-k,q}(x,q^ky), & \text{if } k\text{: odd.} \end{cases}$$

$$(ii)\quad D^{(k)}_{q,y}\,{}_S\,G_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}}\dfrac{[n]_q!}{[n-k]_q!}\,{}_SG_{n-k,q}(x,q^ky), & \text{if } k\text{: even,} \\[2ex] (-1)^{\frac{k-1}{2}}\dfrac{[n]_q!}{[n-k]_q!}\,{}_CG_{n-k,q}(x,q^ky), & \text{if } k\text{: odd.} \end{cases}$$

Proof. (i) Applying the q-derivative in ${}_CG_{n,q}(x,y)$ with respect to y, we obtain

$$\begin{aligned} &D^{(1)}_{q,y}\sum_{n=0}^{\infty}{}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}{}_SG_{n,q}(x,qy)\frac{t^{n+1}}{[n]_q!} \\ &= \sum_{n=0}^{\infty}[n]_q\,{}_SG_{n,q}(x,qy)\frac{t^n}{[n]_q!}. \end{aligned} \tag{2}$$

Using the coefficient comparison method and induction in (2), we can write:

$$\begin{aligned}
D_{q,y}^{(1)}{}_CG_{n,q}(x,y) &= [n]_q{}_SG_{n-1,q}(x,qy)\\
&= \frac{[n]_q!}{[n-1]_q!}{}_SG_{n-1,q}(x,qy),\\
D_{q,y}^{(2)}{}_CG_{n,q}(x,y) &= -[n]_q[n-1]_q{}_CG_{n-2,q}(x,q^2y)\\
&= -\frac{[n]_q!}{[n-2]_q!}{}_CG_{n-2,q}(x,q^2y),\\
&\vdots
\end{aligned}$$

to derive the desired result.

(ii) If we apply the proof process of (i) of Theorem 2.2 similarly to ${}_SG_{n,q}(x,y)$, we can derive (ii) of the theorem; hence, the proof process is omitted. □

Theorem 2.3. *(i) The q-difference equation of the form*

$$\begin{aligned}
&\frac{G_{n,q}}{[n]_q!}D_{q,x}^{(n)}S_{n,q}(x,y)+\frac{G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}S_{n,q}(x,y)\\
&+\frac{G_{n-2,q}}{[n-2]_q!}D_{q,x}^{(n-2)}S_{n,q}(x,y)+\cdots+\frac{G_{2,q}}{[2]_q!}D_{q,x}^{(2)}S_{n,q}(x,y)\\
&+G_{1,q}D_{q,x}^{(1)}S_{n,q}(x,y)+G_{0,q}S_{n,q}(x,y)-{}_SG_{n,q}(x,y)=0
\end{aligned}$$

has ${}_SG_{n,q}(x,y)$ as a solution.

(ii) The polynomial ${}_CG_{n,q}(x,y)$ is a solution of

$$\begin{aligned}
&\frac{G_{n,q}}{[n]_q!}D_{q,x}^{(n)}C_{n,q}(x,y)+\frac{G_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}C_{n,q}(x,y)\\
&+\frac{G_{n-2,q}}{[n-2]_q!}D_{q,x}^{(n-2)}C_{n,q}(x,y)+\cdots+\frac{G_{2,q}}{[2]_q!}D_{q,x}^{(2)}C_{n,q}(x,y)\\
&+G_{1,q}D_{q,x}^{(1)}C_{n,q}(x,y)+G_{0,q}C_{n,q}(x,y)-{}_CG_{n,q}(x,y)=0.
\end{aligned}$$

Proof. (i) Using the generating function of QSG polynomials, we find a relation for ${}_SG_{n,q}(x,y)$, $G_{n,q}$, and $S_{n,q}(x,y)$ as

$$\sum_{n=0}^{\infty}{}_SG_{n,q}(x,y)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}S_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!}. \tag{3}$$

Comparing both sides of Equation (3) for t^n yields,

$$ {}_{S}G_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q} S_{n-k,q}(x,y). \tag{4} $$

If we replace Equation (4) with Theorem 1.5.(i), we can write

$$ {}_{S}G_{n,q}(x,y) = \sum_{k=0}^{n} \frac{G_{k,q}}{[k]_q!} D_{q,x}^{(k)} S_{n,q}(x,y). \tag{5} $$

We obtain the desired result by expanding the series in Equation (5).

(ii) Using a procedure similar to Equation (3) for the QCG polynomial, we can write:

$$ {}_{C}G_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q} C_{n-k,q}(x,y). \tag{6} $$

Using Theorem 1.5.(ii), Equation (6) becomes Equation (7):

$$ {}_{C}G_{n,q}(x,y) = \sum_{k=0}^{n} \frac{G_{k,q}}{[k]_q!} D_{q,x}^{(k)} C_{n,q}(x,y). \tag{7} $$

From Equation (7), we can derive Theorem 2.3. □

Corollary 2.4. *For $q \to 1$ in Theorem 2.3, the following holds:*

(i)
$$ \begin{aligned} &\frac{G_n}{n!}\frac{d^n}{dx^n} S_n(x,y) + \frac{G_{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}} S_n(x,y) \\ &+ \frac{G_{n-2}}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}} S_n(x,y) + \cdots + \frac{G_2}{2!}\frac{d^2}{dx^2} S_n(x,y) \\ &+ G_1 \frac{d}{dx} S_n(x,y) + G_0 S_n(x,y) - {}_{S}G_n(x,y) = 0, \end{aligned} $$

(ii)
$$ \begin{aligned} &\frac{G_n}{n!}\frac{d^n}{dx^n} C_n(x,y) + \frac{G_{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}} C_n(x,y) \\ &+ \frac{G_{n-2}}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}} C_n(x,y) + \cdots + \frac{G_2}{2!}\frac{d^2}{dx^2} C_n(x,y) \\ &+ G_1 \frac{d}{dx} C_n(x,y) + G_0 C_n(x,y) - {}_{C}G_n(x,y) = 0. \end{aligned} $$
□

Theorem 2.5. *Let n be a non-negative integer. Then, the q-difference equation below, for variable y, has ${}_{S}G_{n,q}(x,y)$ as the solution.*

(i) *If n is a even number, then*

$$\begin{aligned}
&\frac{(-1)^{\frac{n}{2}}G_{n,q}}{[n]_q!}D_{q,y}^{(n)}S_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n}{2}}G_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}C_{n,q}(x,q^{1-n}y)\\
&+\frac{(-1)^{\frac{n-2}{2}}G_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}S_{n,q}(x,q^{2-n}y)+\cdots\\
&\cdots-\frac{G_{2,q}}{[2]_q!}D_{q,y}^{(2)}S_{n,q}(x,q^{-2}y)\\
&-G_{1,q}D_{q,y}^{(1)}C_{n,q}(x,q^{-1}y)+G_{0,q}S_{n,q}(x,y)-{}_SG_{n,q}(x,y)=0.
\end{aligned}$$

(ii) *If n is a odd number, then*

$$\begin{aligned}
&\frac{(-1)^{\frac{n+1}{2}}G_{n,q}}{[n]_q!}D_{q,y}^{(n)}C_{n,q}(x,q^{-n}y)\\
&+\frac{(-1)^{\frac{n-1}{2}}G_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}S_{n,q}(x,q^{1-n}y)\\
&+\frac{(-1)^{\frac{n-1}{2}}G_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}C_{n,q}(x,q^{2-n}y)+\cdots\\
&\cdots-\frac{G_{2,q}}{[2]_q!}D_{q,y}^{(2)}S_{n,q}(x,q^{-2}y)\\
&-G_{1,q}D_{q,y}^{(1)}C_{n,q}(x,q^{-1}y)+G_{0,q}S_{n,q}(x,y)\\
&-{}_SG_{n,q}(x,y)=0.
\end{aligned}$$

Proof. In Theorem 1.6.(i), we can formulate

$$S_{n-k,q}(x,y)=\begin{cases}(-1)^{\frac{k}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}S_{n,q}(x,q^{-k}y), & \text{if } k\text{: even},\\[2ex] (-1)^{\frac{k+1}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}C_{n,q}(x,q^{-k}y), & \text{if } k\text{: odd}.\end{cases} \tag{8}$$

Applying Equation (8) in Equation (4), we can complete the proof of Theorem 2.5. □

Corollary 2.6. *Setting $q\to 1$ in Theorem 2.5, the following holds:*

(i) *If n is a even number, then*

$$\frac{(-1)^{\frac{n}{2}}G_n}{n!}\frac{d^n}{dy^n}S_n(x,y)+\frac{(-1)^{\frac{n}{2}}G_{n-1}}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}C_n(x,y)$$
$$+\frac{(-1)^{\frac{n-2}{2}}G_{n-2}}{(n-2)!}\frac{d^{n-2}}{dy^{n-2}}S_n(x,y)+\cdots$$
$$\cdots-\frac{G_2}{2!}\frac{d^2}{dy^2}S_n(x,y)-G_1\frac{d}{dy}C_n(x,y)+G_0S_n(x,y)$$
$$-{}_SG_n(x,y)=0.$$

(ii) *If n is a odd number, then*

$$\frac{(-1)^{\frac{n+1}{2}}G_n}{n!}\frac{d^n}{dy^n}C_n(x,y)+\frac{(-1)^{\frac{n-1}{2}}G_{n-1}}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}S_n(x,y)$$
$$+\frac{(-1)^{\frac{n-1}{2}}G_{n-2}}{(n-2)!}\frac{d^{n-2}}{dy^{n-2}}C_n(x,y)+\cdots$$
$$\cdots-\frac{G_2}{2!}\frac{d^2}{dy^2}S_n(x,y)-G_1\frac{d}{dy}C_n(x,y)+G_0S_n(x,y)$$
$$-{}_SG_n(x,y)=0.$$

□

Theorem 2.7. *For variable y, ${}_CG_{n,q}(x,y)$ is one of the following solutions of the q-difference equations:*

(i) *If n is a even number, then*

$$\frac{(-1)^{\frac{n}{2}}G_{n,q}}{[n]_q!}D_{q,y}^{(n)}C_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n-2}{2}}G_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}S_{n,q}(x,q^{1-n}y)$$
$$+\frac{(-1)^{\frac{n-2}{2}}G_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}C_{n,q}(x,q^{2-n}y)+\cdots-\frac{G_{2,q}}{[2]_q!}D_{q,y}^{(2)}C_{n,q}(x,q^{-2}y)$$
$$+G_{1,q}D_{q,y}^{(1)}S_{n,q}(x,q^{-1}y)+G_{0,q}C_{n,q}(x,y)-{}_CG_{n,q}(x,y)=0.$$

(ii) *If n is a odd number, then*

$$\frac{(-1)^{\frac{n-1}{2}}G_{n,q}}{[n]_q!}D_{q,y}^{(n)}S_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n}{2}}G_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}C_{n,q}(x,q^{1-n}y)$$
$$+\frac{(-1)^{\frac{n-3}{2}}G_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}S_{n,q}(x,q^{2-n}y)+\cdots-\frac{G_{2,q}}{[2]_q!}D_{q,y}^{(2)}C_{n,q}(x,q^{-2}y)$$
$$+G_{1,q}D_{q,y}^{(1)}S_{n,q}(x,q^{-1}y)+G_{0,q}C_{n,q}(x,y)-{}_CG_{n,q}(x,y)=0.$$

Proof. In Theorem 1.6.(ii), it can be observed that

$$C_{n-k,q}(x,y)=\begin{cases}(-1)^{\frac{k}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}C_{n,q}(x,q^{-k}y), & \text{if } k\text{: even,}\\ \\ (-1)^{\frac{k-1}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}S_{n,q}(x,q^{-k}y), & \text{if } k\text{: odd.}\end{cases} \tag{9}$$

Considering Equation (9) in Equation (6), we obtain the result of Theorem 2.7. □

Theorem 2.8. *For $e_q(t)\neq -1$, the QSG polynomial is one of the solutions of the following n-th order difference equation:*

$$\begin{aligned}&\frac{1}{[n]_q!}D_{q,x}^{(n)}{}_SG_{n,q}(x,y)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}{}_SG_{n,q}(x,y)\\&+\frac{1}{[n-2]_q!}D_{q,x}^{(n-2)}{}_SG_{n,q}(x,y)+\cdots\\&+\frac{1}{[2]_q!}D_{q,x}^{(2)}{}_SG_{n,q}(x,y)+D_{q,x}^{(1)}{}_SG_{n,q}(x,y)\\&+2\left({}_SG_{n,q}(x,y)-S_{n,q}(x,y)\right)=0.\end{aligned}$$

Proof. If $e_q(t)\neq -1$ in the generating function of QSG polynomials, the following derivation is obtained:

$$\begin{aligned}&2\sum_{n=0}^{\infty}S_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_SG_{n-k,q}(x,y)+{}_SG_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{10}$$

After comparing the series on both sides in Equation (10), we can write:

$$2S_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_SG_{n-k,q}(x,y)+{}_SG_{n,q}(x,y). \tag{11}$$

If we substitute Theorem 2.1.(ii) into the right-hand side of Equation (10), we can formulate

$$\sum_{k=0}^{n}\frac{1}{[k]_q!}D_{q,x}^{(k)}{}_SG_{n,q}(x,y)+{}_SG_{n,q}(x,y)-2S_{n,q}(x,y)=0. \tag{12}$$

By expanding the finite series on the left-hand side of Equation (12), we obtain the desired result. □

Theorem 2.9. *The q-difference equation*

$$\begin{aligned}&\frac{1}{[n]_q!}D^{(n)}_{q,x}{}_CG_{n,q}(x,y)+\frac{1}{[n-1]_q!}D^{(n-1)}_{q,x}{}_CG_{n,q}(x,y)\\&+\frac{1}{[n-2]_q!}D^{(n-2)}_{q,x}{}_CG_{n,q}(x,y)+\cdots\\&\cdots+\frac{1}{[2]_q!}D^{(2)}_{q,x}{}_CG_{n,q}(x,y)+D^{(1)}_{q,x}{}_CG_{n,q}(x,y)\\&+2\left({}_CG_{n,q}(x,y)-C_{n,q}(x,y)\right)=0\end{aligned}$$

has ${}_CE_{n,q}(x,y)$ as the solution.

Proof. Similar to the procedure used for finding Equation (11) in Theorem 2.8, the relationship between ${}_CG_{n,q}(x,y)$ and $C_{n,q}(x,y)$ is:

$$2C_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_CG_{n-k,q}(x,y)+{}_CG_{n,q}(x,y). \tag{13}$$

Substituting (ii) of Theorem 2.1. into the right-hand side of Equation (13), we obtain:

$$\sum_{k=0}^{n}\frac{1}{[k]_q!}D^{(k)}_{q,x}{}_CG_{n,q}(x,y)+{}_CG_{n,q}(x,y)-2C_{n,q}(x,y)=0. \tag{14}$$

Using Equation (14), we can finish the proof of Theorem 2.9. □

Corollary 2.10. *For $q\to 1$ in Theorems 2.8 and 2.9, the following holds:*

$$\text{(i)}\quad \frac{1}{n!}\frac{d^n}{dx^n}{}_SG_n(x,y)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}{}_SG_n(x,y)$$
$$+\frac{1}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}{}_SG_n(x,y)+\cdots$$
$$\cdots+\frac{1}{2!}\frac{d^2}{dx^2}{}_SG_n(x,y)+\frac{d}{dx}{}_SG_n(x,y)$$
$$+2\left({}_SG_n(x,y)-S_n(x,y)\right)=0.$$
$$\text{(ii)}\quad \frac{1}{n!}\frac{d^n}{dx^n}{}_CG_n(x,y)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}{}_CG_n(x,y)$$
$$+\frac{1}{(n-2)!}\frac{d^{n-2}}{dx^{n-2}}{}_CG_n(x,y)+\cdots$$
$$\cdots+\frac{1}{2!}\frac{d^2}{dx^2}{}_CG_n(x,y)+\frac{d}{dx}{}_CG_n(x,y)$$
$$+2\left({}_CG_n(x,y)-C_n(x,y)\right)=0.$$

□

3. Conclusion

We have identified several differential equations whose solutions are Genocchi polynomials combined with trigonometric functions. It was confirmed that differential equations appear in various ways depending on the variables, and in order to present mathematical modeling in the future, we need to further study various differential equations.

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